Lesson Pre 5.1: Properties of Exponents

You know, we mathematicians are kind of lazy, at least when it comes to writing. If we can find a shorter way of writing something, we're all over it. Take for instance the following expression:

$$x \cdot x \cdot x \cdot y \cdot x \cdot y \cdot y \cdot y \cdot y \cdot x \cdot y \cdot x \cdot x \cdot x$$

What a pain to write out! Man I wish there was a shorter way to write the same thing! Hey, wait a minute! Isn't this exactly what exponents were made for?

What's an exponent?

What's an ***exponent*** you ask? It's that little number in superscript just to the right and above another number. The number on the bottom is kind of like the ***base*** that holds up the exponent. The little number is the count of times the ***base*** is used as a factor (how many times it appears in a multiplication statement).

For instance, 2^3 means $2 \cdot 2 \cdot 2$ …the base is 2 and the exponent is 3.

So how can we use exponents to make writing the mess above simpler? Well, we have *x*'s and *y*'s all multiplied together; the base numbers are *x* and *y*. Since they are all multiplied, we can rearrange them in any order (associative property, order doesn't matter) and count them up. Let's see, we have 8 *x*'s and 6 *y*'s. That means the exponent for the *x* will be 8 and the exponent for the *y* will be 6. We have $x^8 y^6$ …wow, that's ***much*** better! ☺

What happens when you combine exponents?

The world is never very simple is it? Stuff happens all at once…more than one thing at a time. Understanding 2^3 is easy…but what about $2^3 \cdot 2^4$? I suppose we could multiply it all out but I wonder if there is a shortcut that would make our lives easier.

First, what does $2^3 \cdot 2^4$ mean? For starters we have the same base for both exponents. Secondly $2^3 = 2 \cdot 2 \cdot 2$ and $2^4 = 2 \cdot 2 \cdot 2 \cdot 2$ …so $2^3 \cdot 2^4$ must mean $(2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2)$. Okay, now count up the twos…we have 7 of them. So $2^3 \cdot 2^4 = 2^7$.

Now look at the exponents at the start and the exponent at the end. How does 3 and 4 relate to 7? Well 3 + 4 = 7. Hey, there's our shortcut! When we have the product (multiplication) of powers on the same base we can simplify by adding the exponents!

In general, if we have a number ***a*** as the base and the numbers ***m*** and ***n*** as exponents we can say $a^m \cdot a^n = a^{m+n}$. Note that we can only do this with common bases.

How about if we divide?

There is another basic situation when working with exponents that is very common…what if we divide numbers with exponents? Suppose we have $\frac{2^3}{2^2}$. Again note that we have common bases. Let's follow the same investigative process. In the numerator we have $2^4 = 2 \cdot 2 \cdot 2 \cdot 2$ and in the denominator $2^3 = 2 \cdot 2 \cdot 2$. This gives us $\frac{2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2}$. Now, remember with division we can cancel terms that are common to both the numerator and denominator. There are 4 two's in the numerator and 3 in the denominator. Cancelling we're left with just a 2 in the numerator and a 1 in the denominator. So $\frac{2^4}{2^3} = \frac{2 \cdot \cancel{2} \cdot \cancel{2} \cdot \cancel{2}}{\cancel{2} \cdot \cancel{2} \cdot \cancel{2}} = \frac{2}{1} = 2 = 2^1$.

The question to ask is how do the exponents at the start (4 and 3) relate to the exponent at the end (1)? That's easy: 4 – 3 = 1!

This gives us a general rule when dividing numbers with exponents: $\frac{a^m}{a^n} = a^{m-n}$.

What are the other ways of combine exponents?

There are 7 basic situations you can encounter when working with exponents. I'm going to list them now as the ***properties of exponents***. You need to memorize them. It really helps to understand how you can come up with each of them. For each rule I will show an example. Spend some time with these…if you understand how they work, you definitely will remember them ***much*** better! Despite what you may think, these ***do*** make sense.

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The properties of exponents

If ***a*** and ***b*** are any real numbers (the bases), and ***m*** and ***n*** are integers (the exponents), then:

1. $a^m \cdot a^n = a^{m+n}$	Product of Powers	$2^3 \cdot 2^2 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 2^{3+2}$
2. $(a^m)^n = a^{m \cdot n}$	Power of a Power	$(2^3)^2 = 2^3 \cdot 2^3 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6 = 2^{3 \cdot 2}$
3. $(a \cdot b)^m = a^m b^m$	Power of Products	$(2 \cdot 3)^2 = (2 \cdot 3)(2 \cdot 3) = 2 \cdot 3 \cdot 2 \cdot 3 = 2^2 \cdot 3^2$
4. $\dfrac{a^m}{a^n} = a^{m-n}$	Quotient of Powers	$\dfrac{2^4}{2^2} = \dfrac{2 \cdot 2 \cdot \cancel{2} \cdot \cancel{2}}{\cancel{2} \cdot \cancel{2}} = \dfrac{2 \cdot 2}{1} = 2^2 = 2^{4-2}$
5. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$	Power of a Quotient	$\left(\dfrac{2}{3}\right)^2 = \dfrac{2}{3} \cdot \dfrac{2}{3} = \dfrac{2 \cdot 2}{3 \cdot 3} = \dfrac{2^2}{3^2}$
6. $a^0 = 1$	Zero Power	…any number raised to the zero'th power is 1
7. $a^{-m} = \dfrac{1}{a^m}$	Negative Power	…flip any number raised to a negative power

Okay, the first five look like they make sense. But what's up with those last two? Bear with me, I'll show you. This is actually kind of cool! ☺

The ***quotient of powers*** property says we just subtract the exponents when we divide (same bases). We're going to use this. In the following sequence I'll start with 2^4 and then each step divide by 2 (or 2^1). Remember, to get from one step to the next, divide by 2…this means we are subtracting one from the numerator's exponent. Here we go…

$2^4 = 2 \cdot 2 \cdot 2 \cdot 2$	*dividing by 2 each step...*
$2^3 = 2 \cdot 2 \cdot 2$	*...is the same as subtracting one from the exponent...*
$2^2 = 2 \cdot 2$	*...each step we're taking away a 2 as we divide...*
$2^1 = 2$	*...now...divide by 2 again...* $2 \div 2 = 1$ *right?*
$2^0 = 1$	*...the zero power property! Okay, divide by 2 again...watch the exponent!*
$2^{-1} = \dfrac{1}{2} = \dfrac{1}{2^1}$	*...now we have the negative power property!*
$2^{-2} = \dfrac{1}{4} = \dfrac{1}{2^2}$	
$2^{-3} = \dfrac{1}{8} = \dfrac{1}{2^3}$	

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If you look closely, you can see where the ***zero power*** and ***negative power*** properties come from. Isn't that cool?

Putting it all together

Let's try out some examples. Evaluate the following:

1. $(-4)(-4)^3$ $\qquad (-4)(-4)^3 = (-4)^1(-4)^3 = (-4)^{1+3} = (-4)^4 = 256$
2. $[(-3)^2]^3$ $\qquad [(-3)^2]^3 = (-3)^{2\cdot 3} = (-3)^6 = 729$
3. $(3^2x^2y)^2$ $\qquad (3^2x^2y)^2 = 3^{2\cdot 2}x^{2\cdot 2}y^{1\cdot 2} = 3^4x^4y^2 = 81x^4y^2$
4. $5^{-4}\cdot 5^3$ $\qquad 5^{-4}\cdot 5^3 = 5^{-4+3} = 5^{-1} = \dfrac{1}{5}$
5. $\left(2^{-3}\right)^2$ $\qquad \left(2^{-3}\right)^2 = 2^{-3\cdot 2} = 2^{-6} = \dfrac{1}{2^6} = \dfrac{1}{64}$
6. $m^7\cdot\dfrac{1}{m^4}$ $\qquad m^7\cdot\dfrac{1}{m^4} = \dfrac{m^7}{m^4} = m^{7-4} = m^3$
7. $\dfrac{8^3\cdot 8^5}{8^9}$ $\qquad \dfrac{8^3\cdot 8^5}{8^9} = \dfrac{8^{3+5}}{8^9} = \dfrac{8^8}{8^9} = 8^{8-9} = 8^{-1} = \dfrac{1}{8}$
8. $\left(\dfrac{5}{6}\right)^{-3}$ $\qquad \left(\dfrac{5}{6}\right)^{-3} = \left(\dfrac{6}{5}\right)^3 = \dfrac{6^3}{5^3} = \dfrac{216}{125}$

Simplify the following expressions:

1. $\dfrac{5x^4y^3}{8x^5}\cdot\dfrac{3x^3y^5}{6y^4}$ $\qquad \dfrac{5x^4y^3}{8x^5}\cdot\dfrac{3x^3y^5}{6y^4} = \dfrac{15x^7y^8}{48x^5y^4} = \dfrac{5}{16}x^{7-5}y^{8-4} = \dfrac{5x^2y^4}{16}$
2. $\dfrac{2x^6y^4}{6x^3}\cdot\dfrac{4x^2y^3}{12y^5}$ $\qquad \dfrac{2x^6y^4}{6x^3}\cdot\dfrac{4x^2y^3}{12y^5} = \dfrac{8x^8y^7}{72x^3y^5} = \dfrac{1}{9}x^{8-3}y^{7-5} = \dfrac{x^5y^2}{9}$